Introduction to Surreal Numbers: Exploring Real, Infinite and Infinitesimal Values and Their Hackenbush Games

Abstract

The surreal number system, discovered by John H. Conway and Donald Knuth, contains real, infinite and infinitesimal numbers, forming a totally ordered field. This paper discusses the construction and values of surreal numbers, focusing on infinite ordinals, infinite and infinitesimal surreals, as well as their realizations as Hackenbush games.

Introduction

Combinatorial game theory (CGT) is one of the richest theories in mathematics, in which games can be algebraically manipulated. Combinatorial games are two-player (L and R) games with no chances; that is, there are defined rules describing the options each player can move from the starting position. A combinatorial game is defined as

$$G = \{ G^L | G^R \} = \{ a,b,c,\ldots | x,y,z,\ldots \}$$

where a,b,c are the available positions in G^L Left may choose and x,y,z are the available positions in G^R Right may choose. A player wins when the other has no possible moves remaining. Combinatorial games can be represented by numbers in the form of $x = \{ x^L | x^R \}$ where all options in set x^L are strictly less than those in set x^R. When given a number, we can determine its corresponding game position. The idea is to analyze and quantify each player's advantage in the game by measuring their number of winning moves.

Hackenbush is a combinatorial game played on different arrangements of connected blue-red line segments. Segments are connected by their endpoints, with at least one connected to the ground line. The players Left and Right can cut blue and red segments, respectively. When a segment is cut, it is deleted from the game along with all other segments no longer connected to the ground via other edges. Each Hackenbush position represents a numerical value. The representation of numbers, such as 0, $\frac{1}{2}$, -1, 2, as Hackenbush games are visualized in Figure 1.

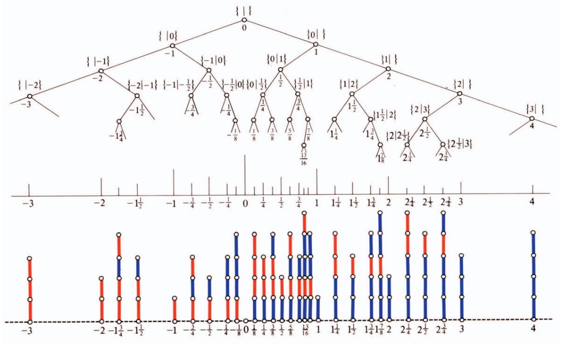

Figure 1. The Number Tree demonstrating the real number line, Hackenbush games and their values, and the Simplicity Rule

Surreal Number System

In this project, we explore the relationship between surreal numbers and their Hackenbush positions. The surreal number system is a totally ordered field which contains the real numbers–the simplest being 0, 1– the infinite (e.g., ω, ω +1) and infinitesimal numbers (e.g., $\frac{1}{\omega}$, $\frac{2}{\omega}$). As Hackenbush can consist of infinitely many segments of various lengths, it is closely tied to surreal values. Through this study, we hope to detail and further the concept of surreal numbers and extend its applications to CGT analysis.

Algebraic Properties of Surreals

Surreal numbers share common arithmetic with real numbers but are derived in the number form of $G = \{G^L | G^R\}$. To obtain the negative of a surreal, we interchange the roles of Left and Right in G. We define the negative of G as

$$-G = \{-G^R | -G^L\}$$

In the context of Hackenbush, the rules for a negative are reversed.

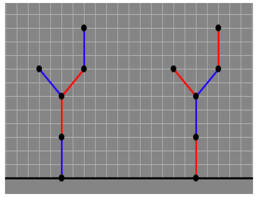

Figure 2. Interchange of segments in Blue-Red Hackenbush game

The addition of surreal numbers or combinatorial games is one of the principal concepts in CGT. The simultaneous playing of two games, G+H, allows Left and Right to choose to move in either game G or H. The sum G+H is defined as

$$G+H = \{ G^L+H, G+H^L | G^R+H, G+H^R \}$$

The results of surreal addition agree with the common arithmetic operation of addition for ordinary real numbers. Similarly, the results of surreal multiplication agree with real ordinary multiplication. Given $x = \{ x^L | x^R \}$ and $y = \{ y^L | y^R \}$, the product is defined as

$$xy = \{ xy^L+x^Ly - x^Ly^L, xy^R+x^Ry - x^Ry^R | xy^R+x^Ly - x^Ly^R, xy^L+x^Ry - x^Ry^L \}$$

Infinite and Infinitesimal Values

In this project, we will illustrate some infinite and infinitesimal surreal values as their Hackenbush positions. While oftentimes infinity ∞ is considered the largest value in mathematics, there are infinite ordinals in the surreal system. The starting infinite number is defined as

$$\omega = \{ 0,1,2,3,\ldots | \}$$

where ω represents the first number greater than all finite numbers. We must note that the surreal number ω is equal to the usual ordinal number ∞; however, its arithmetic differs. For instance, ω-1 = {0,1,2,3,... | ω } does not equal ω, nor does ω + 1.

Infinite Ordinal Hackenbush Positions

The smallest infinite ordinal defined as ω = { 0,1,2,3,... | } has a Hackenbush position where a blue stalk of length 1 is topped by a blue stalk of length $\frac{1}{2}$, followed by length $\frac{1}{4}$, $\frac{1}{8}$ and so on. The negative of it, defined as follows,

$$-\omega = \{| 0, -1,-2,-3,-4,\ldots \}$$

has a similar Hackenbush game constructed of red stalks where player Right has the advantage of ω free moves.

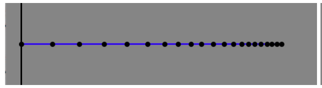

Figure 3. The game ω in Hackenbush

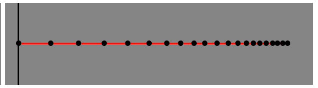

Figure 4. The game -ω in Hackenbush

For infinite ordinal numbers, we can continue to perform transfinite induction to generate greater numbers beyond ω. The first such ordinal greater than ω, ω+1, can be created, and so on. Thus, infinite ordinal numbers, such as ω+1, ω+2, ω^2, and $\sqrt{\omega}$ exist.

To find the value of ω+1, we can use the addition formula to find

$$\omega+1 = \{ \omega | \}$$

where its Hackenbush position can be constructed simply by adding a blue stalk of length 1 on top of the ω tree. The +1 component represents that, on top of ω, Left has an advantage of one more free move.

Provided that we know ω = { 0,1,2,3,... | }, by performing transfinite induction, we can extend the ω tree, produce more transfinite numbers using addition and multiplication, and generate their Hackenbush positions.

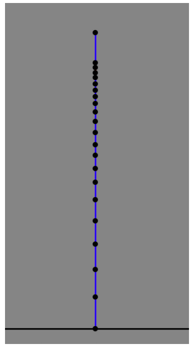

Figure 5. The Hackenbush game ω+1

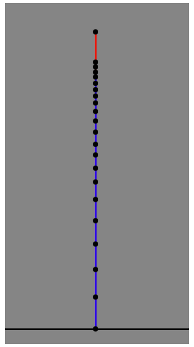

Figure 6. The Hackenbush game ω-1

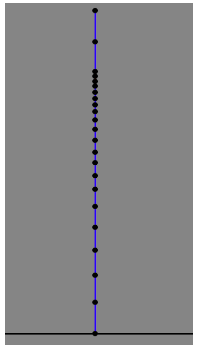

Figure 7. The Hackenbush game ω+2

From the examples above, we can recursively define the following for infinite surreal numbers.

$$\omega + n = \{ \omega + (n-1) | \}$$

$$\omega - n = \{ \mathbb{Z} | \omega - (n-1) \}$$

Infinite Ordinal Multiplication

Using surreal multiplication, we can calculate and generalize more complex surreal values and their respective Hackenbush games.

$$n\omega = \{ (n-1)\omega + \mathbb{Z} | \}$$

$$\frac{\omega}{2\text{^}(n+1)} = \{ \mathbb{Z} | \frac{\omega}{2\text{^}n} - \mathbb{Z} \}$$

Figure 8. The Hackenbush game 2ω

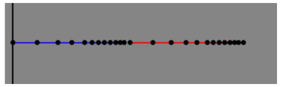

Figure 9. The Hackenbush game $\frac{\omega}{2}$

Infinitesimal Numbers

Having previously defined ω = { ℤ | } and some other infinite ordinals, we can generate the reciprocal of ω, $\frac{1}{\omega}$ or ε, and other infinitesimal values by means of arithmetic operations. The surreal ε is the smallest possible positive number in the surreal system: it is greater than 0, but less than all positive dyadic rationals. Given ω = { ℤ | } , we define

$$\varepsilon = \frac{1}{\omega} = \{ 0 | 1, \tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{8}, \ldots \}$$

Its Hackenbush game consists of a blue stalk of length 1 connected to the ground since the Left set has one element 0, topped by red stalks of length 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$ as it is in the Right set.

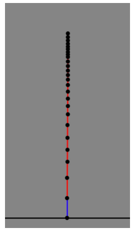

Figure 10. The Hackenbush game ε

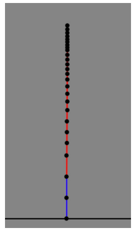

Figure 11. The Hackenbush game ε+1

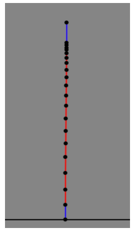

Figure 12. The Hackenbush game 2ε

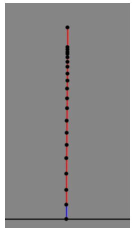

Figure 13. The Hackenbush game $\frac{\varepsilon}{2}$

Conclusion and Future Research

In this study, we introduced the new concept of surreal numbers, specifically on their construction and distinct arithmetic operations. We also calculated and generalized infinite and infinitesimal surreal numbers and represented them as Hackenbush positions. However, there are many other larger and smaller surreal numbers and ordinals that this project has yet to explore. Some values to be considered and potentially solved as Hackenbush positions in future researches are larger ordinals, such as $\omega^{2\omega}$ and $\omega^{\omega+1}$, and more complicated surreal constructions such as $\sqrt[\omega]{\omega}$, $\frac{1}{\omega^\omega}$, and $\frac{1}{\omega^{\sqrt{\omega}}}$. Their corresponding values and Hackenbush positions could be presented in more complex fashions that may require the use of multivariable calculus.